

# Characterization of a subclass of Tweedie distributions by a property of generalized stability

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## Abstract

We introduce a class of distributions originating from an exponential family and having a property related to the strict stability property. Specifically, for two densities  $p_\theta$  and  $p$  linked by the relation

$$p_\theta(x) = e^{\theta x} c(\theta) p(x),$$

we assume that their characteristic functions  $f_\theta$  and  $f$  satisfy

$$f_\theta(t) = f^{\alpha(\theta)}(\beta(\theta)t) \quad \forall t \in \mathbb{R}, \theta \in [a, b], \text{ with } a \text{ and } b \text{ s.t. } a \leq 0 \leq b.$$

A characteristic function representation for this family is obtained and its properties are investigated. The proposed class relates to stable distributions and includes Inverse Gaussian distribution and Levy distribution as special cases.

Due to its origin, the proposed distribution has a sufficient statistic. Besides, it combines stability property at lower scales with an exponential decay of the distribution's tail and has an additional flexibility due to the convenient parametrization. Apart from the basic model, certain generalizations are considered, including the one related to geometric stable distributions.

**Key Words:** Natural exponential families, stability-under-addition, characteristic functions.

## 1 Introduction

Random variables with the property of stability-under-addition (usually called just *stability*) are of particular importance in applied probability theory and statistics. The significance of *stable laws* and related distributions is due to their major role in the Central limit problem and their link with applied stochastic models used in e.g. physics and financial mathematics.

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In practical applications, stable laws appearing naturally as limit distributions for sums of random variables (the essence of the Central limit problem) are often used to describe the increments of stochastic processes. Stable distributions may exhibit a reliable fit of empirical data whose nature is related to random summation. However, certain peculiarities of the data (e.g., lighter tails than according to stable models) motivate the search for alternatives to stable laws. Among different ways to adjust the tail behavior of the stable distribution is the exponential (tilting), the smoothening of the tail of stable density is widely applied, see e.g. [14].

In the present work, we consider exponential families with a property that can be considered as a generalization of the usual strict stability-under-addition property (that is, with a real-valued function in place of the natural number of summands). The result is, as appears, related to exponentially tilted stable distributions.

As the class we consider originates from natural exponential families and related to stable distributions, it is not surprising that our class appears to be a subclass of Tweedie distributions, that, for some combination of parameters, can also be characterized as exponentially tilted stable distributions.

Therefore, the result of our findings is a convenient representation of a certain subclass of Tweedie distributions – a characterization in terms of extended stability property. Like it is for the Tweedie distributions, the important advantage of the proposed class is the existence of a sufficient statistics which is due to its belonging to natural exponential families, - and the combination of the generalized stability with exponential tails.

## 2 Definition of the class, derivation and properties of its chf

Suppose  $X$  is a real random variable with cdf  $P$  and pdf  $p$ . A class of distributions associated with  $P$  is the *natural exponential family* with the pdf

$$p_\theta(x) = e^{\theta x} c(\theta) p(x), \quad (1)$$

with the real parameter  $\theta \in [a, b]$ , where  $a$  and  $b$  are such that  $0 \in [a, b]$ . With  $f_\theta(t)$  denoting the characteristic function (chf) of  $p_\theta(x)$  and  $f(t)$  the chf of  $p(x)$ , the following equality is valid according to (1):

$$f_\theta(t) = \frac{f(t - i\theta)}{f(-i\theta)}, \quad \theta \in [a, b]. \quad (2)$$

The problem we investigate in relation to the above chf is the following: Do real valued functions  $\alpha(\theta)$  and  $\beta(\theta)$  exist, such that

$$f_\theta(t) = f^{\alpha(\theta)}(\beta(\theta)t) \quad \forall t \in \mathbb{R}, \quad \theta \in [a, b]. \quad (3)$$

It is known (see e.g. [11]) that if a chf is analytic in the neighborhood of the origin, then it is also analytic in a horizontal strip (either this strip is the whole plane or it has one or two horizontal boundary lines). It follows from the above that  $\forall \theta \in [a, b]$ , chf  $f_\theta(t)$  is analytic ( $t \in \mathbb{C}$ ) in the horizontal strip  $|\operatorname{Im}(t)| < \rho(\theta)$  for some  $\rho(\theta) > 0$ . Then with the notation  $g(t) := \ln(f(t))$ , one gets from (2) and (3) the following balance equation

$$g(t - i\theta) = g(-i\theta) + \alpha(\theta)g(\beta(\theta)t). \quad (4)$$

The above functional equation can be solved using an argument based on sequential differentiating w.r.t. to  $\theta$  and to  $t$ , which we postpone to Appendix. The solution w.r.t. the function  $g(t)$  has the form

$$g(t) = A[1 - (1 - itc)^\gamma], \quad (5)$$

where  $\gamma \leq 2$ ,  $A, c \in \mathbb{R}$  and the sign of  $A$  depends on the value of  $\gamma$  (as we shall see,  $A < 0$  when  $\gamma < 0$  and when  $\gamma = 2$ ;  $A > 0$  when  $\gamma \in (0, 1)$  and when  $\gamma \in (1, 2)$ ). The expression for the chf  $f(t)$  immediately follows

$$f(t) = \exp \{A[1 - (1 - itc)^\gamma]\}.$$

Recalling (2), we can now write down the chf  $f_\theta$

$$f_\theta(t) = \exp \{A[(1 - c\theta)^\gamma - (1 - itc - c\theta)^\gamma]\} \quad (6)$$

$$\begin{aligned} &= \exp \left\{ A \cdot (1 - c\theta)^\gamma \left[ 1 - \left( 1 - \frac{itc}{1 - c\theta} \right)^\gamma \right] \right\} \\ &= \exp \left\{ A \cdot B^\gamma \left[ 1 - \left( 1 - \frac{itc}{B} \right)^\gamma \right] \right\}, \end{aligned} \quad (7)$$

where  $B := 1 - c\theta$ .

Note that the functions  $\alpha(\theta)$  and  $\beta(\theta)$  are then explicitly expressed

$$\alpha(\theta) = (1 - c\theta)^\gamma, \quad \beta(\theta) = 1/(1 - c\theta).$$

## 2.1 Ranges of the shape exponent $\gamma$ ; role of parameters $A$ and $c$

The properties of introduced distributions differ with respect to various combinations of their parameters  $\gamma, A, c$ .

As the role of the shape exponent  $\gamma$  is crucial to relations between other parameters, in the present paragraph we analyze different cases that arise with respect to certain ranges of  $\gamma$ . That classification w.r.t.  $\gamma$  also makes it easier to clarify the relation of the introduced class to stable laws and other important distributions.

Returning to the representation for the chf  $f$

$$f_\gamma(t) = \exp \{A[1 - (1 - itc)^\gamma]\}, \quad (8)$$

note that the properties of the corresponding distribution differ depending on the range of  $\gamma$  (to mark the importance of this dependency, we shall use the notation  $f_\gamma$ , instead of just  $f$ , and will point out the range of  $\gamma$ , such as  $f_{\gamma < 0}$  or  $f_{\gamma \in (1, 2)}$ ). Listing different cases w.r.t.  $\gamma$  below, we shall see that for each of the ranges of  $\gamma$ ,  $f_\gamma$  is a proper chf only for a particular sign of  $A$ . In all cases, there exists a corresponding analytic function  $f_\gamma(t)$ ,  $t \in \mathbb{C}$ , that agrees with the characteristic function  $f_\gamma(t)$ ,  $t \in \mathbb{R}$ . The domain of regularity for corresponding analytic function varies depending on the ranges of  $\gamma$  considered below.

**Case a:**  $\gamma < 0$  Denoting  $\bar{\gamma} = -\gamma$ , for this case we have

$$f_{\gamma < 0}(t) = \exp \{A[1 - (1 - itc)^{-\bar{\gamma}}]\}. \quad (9)$$

As  $\hat{\gamma}(t) := (1 - itc)^{-\bar{\gamma}}$  is the chf of a gamma distribution, we see that  $f_{\gamma < 0}$  corresponds to the chf of the compound Poisson r.v. with gamma-distributed summands, provided that  $\bar{A} := -A > 0$ :

$$f_{\gamma < 0}(t) = \exp \{\bar{A}[\hat{\gamma}(t) - 1]\}. \quad (10)$$

In other words,  $f_{\gamma < 0}(t)$  is a chf of the random sum  $S = \sum_{i=1}^{N(t)} X_i$ , where  $N \sim \text{Poisson}$  and  $X_1 \sim \text{gamma}(\bar{\gamma}, c)$ , i.e.  $\bar{\gamma}$  and  $c$  are the shape and the scale parameter correspondingly.

Clearly,  $f_{\gamma < 0}(t)$ ,  $t \in \mathbb{C}$ , is analytic when  $\text{Im}(t) \in (-1/c, \infty)$ , with the entire line  $t = -i/c$  being the region of singularity for  $f(t)$ .

**Case b:**  $\gamma \in (0, 1)$  This case, after a complex shift of the variable, corresponds to a certain subclass of stable distributions. Specifically, recall that if  $\tilde{X} = X + x_0$  is a *shifted one-sided Levy-stable* variable then its chf can be represented as

$$\tilde{\phi}(z) = \exp \{ izx_0 - (-iz)^{\alpha} \tilde{a} \}, \quad \tilde{a} > 0. \quad (11)$$

Clearly, (11) with  $x_0 = 0$  brings us close to our case (8), differing by just a proper change of the variable. Specifically, with  $cz := ct + i$  we have

$$f_{\gamma \in (0,1)}(t) = e^A \exp \{ -(1 - itc)^{\gamma} A \} \iff f_{\gamma \in (0,1)}(z - i/c) = e^A \exp \{ -(-izc)^{\gamma} A \}, \quad (12)$$

which is close to (11) up to the normalization factor  $e^A$  (provided that  $A > 0$ ).

Furthermore, note that the re-parametrization  $cz = ct + i$  relates to the transformation of the density, similar to the one used in the introduction of  $f_{\theta}$ , according to (1) and (2). When  $t = z - i/c$  then in order for  $f_{\gamma \in (0,1)}(t)$  to correspond to a proper chf we need a normalization factor  $f_{\gamma \in (0,1)}(-i/c)$ . From (12), we see that  $\frac{f_{\gamma \in (0,1)}(z - i/c)}{f_{\gamma \in (0,1)}(-i/c)} =: f_{\gamma}^{stab}(z)$  corresponds to one-sided stable chf and that  $f_{\gamma \in (0,1)}(-i/c) = e^A$ .

Hence, one-sided stable density corresponding to chf  $f_{\gamma}^{stab}$  is the result of an exponential transformation (take the exponent parameter  $1/c$  in (1) instead of  $\theta$ ) of the density corresponding to chf  $f_{\gamma \in (0,1)}$ . That is, the r.v. with chf  $f_{\gamma \in (0,1)}$  in turn relates to the exponential transformation of the one-sided stable density with the exponent parameter being  $-1/c$ . Note that both  $f_{\gamma}^{stab}$  and  $f_{\gamma \in (0,1)}$  are proper characteristic functions only when  $A > 0$ .

Like in Case a, the function  $f_{\gamma \in (0,1)}(t)$ ,  $t \in \mathbb{C}$ , is analytic in  $\text{Im}(t) \in (-1/c, \infty)$ , but contrary to the previous case, there exists a limit for  $f_{\gamma \in (0,1)}(t)$  w.r.t  $t$  approaching the line  $t = -i/c$  (with the only point of non-regularity being  $(0, -i/c)$ ). This limit corresponds to  $f_{\gamma}^{stab}$  introduced above which is the chf of one-sided stable law.

Note that the above is in correspondence with one of the classical results on characteristic functions that says that a necessary condition for a function analytic in some neighborhood of the origin to be a characteristic function is that in either half-plane the singularity nearest to the real axis is located on the imaginary axis (see e.g. [11] for details).

**Remark 1** Check that the well-known Inverse Gaussian distribution as a special case with  $\gamma = 1/2$ .

**Case c:**  $\gamma = 1$  This degenerate case corresponds to the constant r.v. whose cdf is the Heaviside step function, since the distribution corresponding to the chf  $e^{iAct}$  has a single unit jump at point  $Ac$ .

**Case d:**  $\gamma \in (1, 2)$  Consider a re-parametrization  $\tilde{\gamma} = \gamma/2$  allowing to write down the chf  $f(t)$  as

$$\tilde{f}_{\gamma \in (1,2)}(t) = \exp \left\{ A[1 - ((1 - itc)^2)^{\tilde{\gamma}}] \right\} = \exp \left\{ A[1 - (1 - ct)^{\tilde{\gamma}}] \right\} = f_{\tilde{\gamma} \in (0,1)}(-it), \quad (13)$$

where  $\tilde{t} := 2it + ct^2$ .

While  $f_{\tilde{\gamma} \in (0,1)}(t)$  is a chf of the type considered above (Case **b**),  $f_{\gamma \in (1,2)}(t)$  is also a chf. Indeed, while  $\phi_{(\mu,\sigma)}(t) = \exp\{\mu it - \sigma^2 t^2/2\}$  is the chf of the Gaussian r.v. (with expectation  $\mu$  and variance  $\sigma^2$ ), the function

$$f_{\tilde{\gamma} \in (0,1)}(-i\tilde{t}) = \int e^{\tilde{t}x} p_{\tilde{\gamma}}(x) dx = \int [\phi_{(2,2\bar{c})}(t)]^x p_{\tilde{\gamma}}(x) dx \quad (14)$$

which is a chf of some probabilistic distribution, as a continuous mixture of the Gaussian chf  $\phi_{(2,2\bar{c})}(t) = \exp\{2it - \bar{c}t^2\}$  (provided that  $\bar{c} := -c > 0$ ) with the pdf  $p_{\tilde{\gamma}}$  whose chf  $f_{\tilde{\gamma} \in (0,1)}$  is the one from the above Case **b**.

Since we already know from Case **b** that it should be that  $A > 0$  for  $f_{\gamma \in (0,1)}$  to be a chf, in the present case we also have  $A > 0$  as a necessary condition for  $f_{\gamma \in (1,2)}$  given by (14) to be a chf. Like in Case **b**,  $f(t)$ ,  $t \in \mathbb{C}$ , is analytic only in the strip restricted by the horizontal line  $t = -i/c$ , i.e. when  $\text{Im}(t) \in (-1/c, \infty)$ , but additionally we have the condition that  $c < 0$ .

**Case e:**  $\gamma = 2$  Check that  $f_{\gamma=2}$  is the chf of the normal r.v.

$$f_{\gamma=2}(t) = \exp\{\bar{A}(-2cit - c^2 t^2)\} = \phi_{(-2\bar{A}c, 2\bar{A}c^2)}(t), \quad (15)$$

where  $\bar{A} = -A$ . As the variance is  $\sigma^2 = 2\bar{A}c^2$ , it should be that  $A < 0$ .

**The sign of  $c$**  Discussing the cases with respect to the parameter  $\gamma$ , we noted that the sign of the parameter  $A$  is crucial in each of the cases. Commenting on the sign of the other parameter,  $c$ , we noted that its sign is only crucial in Case **d**. In other cases,  $c$  can take any sign when  $f_{\gamma}$  is a proper chf due to the property of characteristic functions providing that  $f(-t)$  is a chf as soon as  $f(t)$  is a chf. However, we should keep in mind that the function  $f_{\gamma}(t)$  ( $t \in \mathbb{C}$ ), in all relevant cases, is analytic only in a strip depending on  $c$ .

**Limit behavior of  $f$  regarding mutual limit w.r.t.  $c$  and  $A$**  Note that the one-sided stable distributions appearing in Case **b** and obtained via the change of the complex variable  $cz := ct + i$  could also be viewed as a limiting case w.r.t  $c \rightarrow \infty$  and  $A \rightarrow 0$ . Additionally,  $c$  and  $A$  should be linked so that  $A \sim \tilde{c}/c^{\gamma}$  ( $\tilde{c} \in \mathbb{R}$ ), hence the limit w.r.t  $A \rightarrow 0$  and  $c \rightarrow \infty$  (keeping  $\tilde{c}$  constant) leads to

$$f_{\gamma \in (0,1)}(t) = e^A \exp\left\{-\left(\frac{1}{c} - it\right)^{\gamma} (c^{\gamma} A)\right\} \xrightarrow[A \rightarrow 0]{c \rightarrow \infty} \exp\{-(-it)^{\gamma} \tilde{c}\} =: f_{\gamma \in (0,1)}^{(lim)}(t),$$

which corresponds to one-sided stable distributions appearing in Case **b** above. Recalling the notation of Case **b**, we see that  $f^{(lim)}(t) = f_{\gamma}^{stab}(t) = \exp\{-(-itc)^{\gamma} A\}$ .

## 2.2 The role of the natural exponent $\theta$

Let us turn again to the representation for  $f_{\theta}$ . Rewrite (6) noting that an additional re-parametrization with  $\tilde{A} := A \cdot B^{\gamma}$  and  $c_{\theta} = \frac{c}{1-c\theta}$  leads to the form (8)

$$f_{\gamma,\theta}(t) = \exp\left\{\tilde{A} \cdot [1 - (1 - itc_{\theta})^{\gamma}]\right\}, \quad (16)$$

where we used the notation  $f_{\gamma,\theta}$  instead of just  $f_\theta$ , indicating the significance of both of the parameters.

Clearly, Cases considered above in relation to chf  $f_\gamma$  can be revisited for  $f_\theta$ , but apart from the ranges of the shape exponent  $\gamma$ , the role the natural exponent  $\theta$  is important.

Obviously, the borders of the analyticity strips depending on  $c$ , as well as the conditions on  $A$ , will change in correspondence with  $\theta$ , i.e. one can view introducing  $f_{\gamma,\theta}$  in place of  $f_\gamma$  as passing from  $\theta = 0$  to non-zero  $\theta$ , so that the "new"  $c$  and  $A$  will depend on  $\theta$ .

In particular, replacements through Case **a** – Case **e** are the following:

- the strips of analyticity:  $(-i/c, \infty) \implies (-i(1/c - \theta), \infty)$  in Cases **a**, **b** and **d**;
- the signs of coefficients: recall that Case **a** and Case **e** imply  $A < 0$ , while in both Case **b** and Case **d** we have  $A > 0$ ; for  $f_{\gamma,\theta}$  the same holds with  $A \implies A \cdot B^\gamma = A \cdot (1 - c\theta)^\gamma$ .

Revisit, for instance, Case **d** where we have a representation analogous to the one expressed by (13) and (14):

$$\tilde{f}_{\gamma \in (1,2),\theta}(t) = f_{\gamma \in (0,1),\theta}(2t - ict^2) = \int [\phi_{(2,2\bar{c}_\theta)}(t)]^x p_{\gamma,\theta}(x) dx, \quad (17)$$

where pdf  $p_{\gamma,\theta}$  corresponds to the chf  $f_{\gamma \in (0,1),\theta} = \exp \left\{ \tilde{A} [1 - (1 - itc_\theta)^{\tilde{\gamma}}] \right\}$  and  $\phi_{(2,2\bar{c}_\theta)}$  is a chf of the normal r.v., i.e.  $\phi_{(2,2\bar{c}_\theta)}(t) = \exp \{ 2it - \bar{c}_\theta t^2 \}$  (with  $\bar{c}_\theta := -c_\theta$ ). So that in order for  $\tilde{f}_{\gamma \in (1,2),\theta}$  to be a chf, it should be that  $c_\theta = c/(1 - c\theta) < 0$  and  $\tilde{A} = A \cdot B^\gamma > 0$ .

All other cases w.r.t  $\gamma$  can be revisited for  $f_{\gamma,\theta}$  analogously. Additionally, certain values and ranges of  $\theta$  correspond to particular special cases which we consider below.

**Range of  $\theta$ , singularities and corresponding limits.** While  $c$  can be positive or negative, the point when  $\theta = 1/c$  is the case of singularity and/or the limiting case.

Specifically, Case **a** with  $\theta = 1/c$  is the case of singularity of the chf  $f_{\gamma,\theta}$ . In Case **b**, the limit  $1/c \leftarrow \theta$  exists with limiting chf's  $f_\gamma^{(lim)}(t) = \exp \{ -(-izc)^\gamma A \}$ .

In a way, approaching the point  $1/c$  with respect to  $\theta$  means approaching corresponding stable (or mixed normal-stable) distributions, while increasing the exponent parameter  $\theta$  corresponds to lightening the tails of distributions  $P_\theta$  to exponential ones.

**Limit case  $\theta \longrightarrow \infty$ .** The question in interest is: what distributions appear in the limit case  $\theta \longrightarrow \infty$ ? In [1], limit laws for the whole class of natural exponential families w.r.t. the growing exponent parameter were investigated. It was shown that for the r.v.  $X_\theta$  with density of the form (1), constants  $a_\theta > 0$  and  $b_\theta$  can be chosen so that a limit  $(X_\theta - b_\theta)/a_\theta \longrightarrow Y$ ,  $\theta \rightarrow \infty$  exists and that only possible limit distributions  $Y$  are the normal and gamma distributions. Specifically, the following results were proved.

- If  $P_\theta$ ,  $\theta \in \Theta$ , is the cdf of the exponential family  $X_\theta$  whose pdf is given by (1) and if there exist constants  $a_\theta > 0$  and  $b_\theta \in \mathbb{R}$  such that  $P_\theta \left( \frac{x - b_\theta}{a_\theta} \right) \xrightarrow{\theta \rightarrow \infty} G$  weakly to some non-degenerate cdf  $G$  then  $G$  belongs to the so called extended gamma family, which includes normal and gamma densities.
- In the case of convergence to a non-degenerate limit  $G$ ,  $a_\theta$  and  $b_\theta$  should be taken such that  $P_\theta$  is centered and scaled by expectation and standard deviation.

Criterion of the convergence of an exponential family (1) to gamma/normal distribution can be expressed in terms of the *moment generating function* (mgf) associated with  $p(x)$ , i.e.  $M(\lambda) = \int e^{\lambda x} p(x) dx$ . Specifically, as proved in [1], the following holds.

*If the function  $s(\lambda) = 1/\sqrt{m''(\lambda)}$ , where  $m(\lambda) := \ln M(\lambda)$ , is self-neglecting, then the exponential family  $X_\lambda$ ,  $\lambda \in \Theta$ , is asymptotically normal. Otherwise, the only possible limit distribution is gamma distribution.*

It is easy to check when the normal distribution appears as a limit in our case. Recalling (5), for the log-mgf (or cumulant generating function) of the introduced family we have

$$m(\lambda) = A[1 - (1 - \lambda c)^\gamma],$$

so that

$$m''(\lambda) = Ac^2\gamma(\gamma - 1) [(1 - \lambda c)^{\gamma-2}].$$

Clearly, the above condition on  $s = 1/\sqrt{m''}$  is only satisfied when  $\gamma \geq 2$ . That means that in all the cases w.r.t. to  $\gamma$  that are relevant for our study, the only possible limit distribution  $G$  is gamma distribution, i.e. according to the usual terminology, our distribution lies in the *domain of attraction* (DA) of gamma distribution.

### 3 Relation to the Tweedie class and special cases

Like briefly noted above, the proposed distributional family includes some well-known models, which can be considered as special cases of exponential family with stability property. At the same time, it itself is subclass of another class – the Tweedie distributions [?], which, in its turn, is a part of even a wider class of *exponential dispersion models*.

#### 3.1 Relation to the Tweedie class

For general definitions and characterizations of *exponential dispersion models* we refer to [?]. The pdf of an exponential dispersion model (EDM) is represented in a way similar to that of our class

$$f_{ED}(y) = p_\beta(y) \exp \left\{ \frac{1}{\beta} (\theta y - \kappa(\theta)) \right\},$$

where  $\beta > 0$ . A convenient representation of the Tweedie class is through the moment generating function  $M(t) = \int \exp(ty) f(y) dy$ . The cumulant generating function of the EDM is

$$m_{ED}(t) = \log M_{ED}(t) = [\kappa(\theta + t\phi) - \kappa(\theta)]/\beta.$$

As the derivatives of the function  $\kappa(\cdot)$  w.r.t  $\theta$  provide the values of the cumulants of the distribution,  $\kappa(\cdot)$  is called the cumulant function. So that we have  $\mu = \kappa'(\theta)$  for the mean and  $\beta\kappa''(\theta)$  for the variance. Since the mapping from  $\theta$  to  $\mu$  is invertible, there exists a function  $V(\cdot)$ , called the variance function, such that  $\kappa''(\theta) = V(\mu)$ .

The Tweedie class is then defined as EDM with the variance function  $V(\cdot)$  of a specific form, namely  $V(\mu) = \mu^p$  for  $p \in (-\infty, 0] \cup [1, \infty)$  and  $V(\mu) = \exp(v\mu)$  for  $p = \infty$  (here,  $v \neq 0$ ). The correspondent cumulant function  $\kappa$  for the Tweedie class can be represented as

$$\begin{cases} \frac{\mu^{2-p}-1}{2-p}, & \text{when } p \neq 2 \\ \log \mu, & \text{when } p = 2. \end{cases} \quad (18)$$



Defining a new parameter  $\alpha$  via

$$(p-1)(1-\alpha) = 1,$$

a representation for the moment generating function of the Tweedie class is obtained:

$$M_{Tw}(s) = \begin{cases} \exp \left\{ \frac{(1-p)^\alpha \theta^\alpha}{\beta(2-p)} [(1+s/\theta)^\alpha - 1] \right\}, & \text{when } p \neq 1, 2, \\ (1+s\theta)^{-1/\beta}, & \text{when } p = 2, \\ \exp \left\{ \frac{1}{\beta} e^\theta (e^s - 1) \right\}, & \text{when } p = 1. \end{cases} \quad (19)$$

From above, comparing with the chf (8) of our class, it is clear that our class is actually a subclass of Tweedie distributions with some parameter  $p \neq 1, 2$  and a certain combination of other parameters.

Correspondingly, the basic properties that we discussed above do hold for the Tweedie class as well. For instance, it is known that the Tweedie distributions can be obtained through exponential damping of stable distributions.

The distributions of the Tweedie class are known to have a *scale invariance property*, that can be expressed through the pdf  $f_{Tw}$  as

$$f_{Tw}(x; \mu, \phi) = c f_{Tw}(cx; c\mu, c^{2-p}\beta), \quad \text{for } c > 0,$$

which also resembles stability property (in terms of scale invariance), however is not directly related to the generalized stability property introduced by considering the relation (3).

Note also that our subclass does not explicitly contain the parameter  $\beta$  in the above ED notation (parameter related to the variance of the distribution).

Besides, note that gamma distribution belonging to the Tweedie class, in our case appears as a limit case  $\theta \rightarrow \infty$ , like mentioned in the previous section.

### 3.2 Inverse Gaussian and Levy distributions

Two important special cases are visible immediately from the form of the characteristic functions  $f_\gamma$  and  $f_{\gamma, \theta}$ .

Recall the characteristic function representation of the well-known *inverse Gaussian distribution*

$$g_{IG}(t) = e^{\frac{\lambda}{\mu} \left[ 1 - \left( 1 - \frac{2\mu^2 it}{\lambda} \right)^{1/2} \right]} \quad (20)$$

and note (as already briefly remarked in Case **b** of Paragraph 2.1) that it can be viewed as a special case of the introduced exponential family with stability — just plug the parameters' values ( $A = \frac{\lambda}{\mu}, c = \frac{2\mu^2}{\lambda}, \gamma = \frac{1}{2}$ ) into the general representation (8).

Recall an important property of the inverse Gaussian distribution:

- If  $X_i$  has an  $IG(\mu w_i, \lambda w_i^2)$  distribution for  $i = 1, 2, \dots, n$  and  $X_i$  are independent, then  $S = \sum_{i=1}^n X_i \sim IG(\mu \bar{w}, \lambda \bar{w}^2)$ , where  $\bar{w} = \sum_{i=1}^n w_i$ . In other words,  $X \sim IG(\mu, \lambda) \Rightarrow tX \sim IG(t\mu, t\lambda)$



Note that this property is related to the scale invariance property and recall that stable distributions are also included in certain subclasses of introduced exponential family (see Case **b** and Case **d**), so that the scale invariance appears to be naturally embedded in our model.

Another well known distribution appears as a limiting case of the Inverse Gaussian distribution: the chf  $f_{Levy}$  of the *Levy distribution* is clearly a limit of  $g_{IG}$  w.r.t.  $\mu \rightarrow \infty$  (and respectively  $\lambda \rightarrow 0$ ) which leads from (20) to

$$f_{Levy}(t) = e^{-\sqrt{-2ict}}. \quad (21)$$

Inverse Gaussian and Levy distributions are the important cases as they have simple explicit forms for their pdf's.

### 3.3 Note on Tempered Levy processes

The combination of the properties of our subclass relates in its purpose to other modified models such as the so-called Tempered Levy process.

The idea behind Tempered Levy processes (also known as Truncated Levy flights [14]) is to use a distribution which coincides with the stable one around zero (i.e. for small fluctuations) and has heavy tails yet decreasing to zero fast enough to assure a finite variance.

The latter is achieved by truncating stable distributions, and one of the ways the truncation can be done is the so called smooth exponential truncation. The result corresponds to an infinitely divisible distribution enabling an analytical chf representation:

$$\phi_{\alpha;\mu}(t) = \exp \left\{ \frac{-a_\alpha}{\cos(\pi\alpha/2)} \left[ (\mu^2 + t^2)^{\alpha/2} \cos \left( \alpha \arctan \left( \frac{|t|}{\mu} - \mu^\alpha \right) \right) \right] \right\},$$

where  $\mu$  is the cut-off parameter. The asymptotic behavior of  $\phi$  is given by

$$\phi_{\alpha;\mu}(t) \underset{t \rightarrow \infty}{\sim} \phi_\alpha(t)$$

where  $\phi_\alpha(t)$  is the chf of the (non-truncated) stable distribution, i.e. for small values of  $x$ , the pdf  $p_{\alpha;\mu}(x)$  corresponding to the chf  $\phi_{\alpha;\mu}$  behaves like a stable law of index  $\alpha$ .

This concept is then interpreted in terms of Levy processes and associated Levy measures. Specifically, if  $\nu(x)$  is a measure of a Levy process (generally, defined as the expected number, per unit time, of jumps whose size is less than  $x$ ), then the measure

$$\tilde{\nu}(dx) = e^{-\theta x} \nu(dx),$$

corresponds to another Levy process, whose large sizes are "tempered" with exponential damping.

The ideas are therefore similar to the one that lead to the introduction of the Tweedie class, as well as to the characterization of our subclass, yet with certain methodological difference: The idea of exponential tempering utilizes stable distribution as the initial distribution whose tail is then exponentially smoothened, while in our approach the stability property appears naturally from the properties of the introduced class. That allows to view the exponential family with stability property as a natural extension of both classes: stable distributions and exponential families which is promising regarding their practical applications, including the use of sufficient statistic naturally inherent in exponential families.

## 4 Modifications related to geometric stable laws

**Preliminaries and definitions** First, recall the origin of *geometric infinitely divisible* (GID) distributions.

The chf of an infinitely divisible (ID) r.v.  $X$  implies the representation  $\phi(t) = (\phi_n(t))^n$ , where the chf  $\phi_n(t)$  corresponds to some r.v.  $X^{(n)}$ , which in turn implies that the sum  $X_1^{(n)} + \dots + X_n^{(n)}$  of iid copies of  $X^{(n)}$  has the same distribution as  $X$ . The so-called *transfer theorems* (see e.g. [3]) state that the random sums

$$X_1^{(n)} + \dots + X_{\nu_n}^{(n)}, \quad (22)$$

where  $(\nu_n)$  is a sequence of integer-valued r.v.'s such that  $\nu_n \xrightarrow{p} \infty$  (in probability) while  $\nu_n/n \xrightarrow{d} \nu$  (in distribution), converge (in distribution) to a r.v.  $Y$  whose chf has the form

$$\omega(t) = \mathcal{L}(-\ln \phi(t)), \quad (23)$$

where  $\mathcal{L}$  is the Laplace transform of the r.v.  $\nu$ .

If  $\nu_n \stackrel{d}{=} \nu_p$  has a geometric distribution with mean  $1/p$  (where  $p$  is close to zero, and  $\nu_p$  is independent of  $X_i'$ s) which converges in distribution to the standard exponential distribution with Laplace transform  $\mathcal{L}(u) = (1 + u)^{-1}$  then (23) turns into

$$\omega(t) = (1 - \ln \phi(t))^{-1}, \quad (24)$$

and random variables whose chf has the form (24) were introduced in [6] as *geometric infinitely divisible* (GID) random variables. Since sums such as (22) frequently appear in many applied problems, GID have a variety of applications. The *geometric stable distributions* (GS), also originated from [6] and developed in later works e.g. [3] and [5], appear as a natural subclass of GID and are widely used for modeling of stochastic processes. The chf of GS laws has the form (24) with  $\phi(t)$  being the chf of a stable r.v.

It is natural to extend the exponential family with stability property introduced in previous sections w.r.t geometric stability. Specifically, for the chf  $f_{\gamma,\theta}$  of the form (16), the corresponding geometric extension will have the chf

$$\omega_{\gamma,\theta}(t) = \frac{1}{1 - \ln f_{\gamma,\theta}(t)}, \quad \forall t \in \mathbb{R}, \quad \theta \in [a, b]. \quad (25)$$

Since  $f_{\gamma,\theta}(t) = \exp \left\{ \tilde{A} \cdot [1 - (1 - itc_\theta)^\gamma] \right\}$ , note that when  $\tilde{A} = A \cdot B^\gamma < 0$  (which corresponds to certain ranges of  $\gamma$ ) then (25) can be viewed as a sum of geometric progression with the initial value  $(1 - \tilde{A})^{-1}$  and common ratio  $\frac{\tilde{A}}{1 - \tilde{A}} (1 - itc_\theta)^\gamma$ .

Revisiting the cases considered in Paragraph 2.1 again, this time in relation to  $\omega_{\gamma,\theta}$ , the argument above leads to a convenient interpretation of Case **a**, which we sketch below, along with other cases w.r.t the ranges of  $\gamma$ .

- **Case a**  $\gamma < 0$ . In this case,  $f_{\gamma<0,\theta}$  is a chf of a probability distribution only when  $\tilde{A} < 0$ , which coincides with the condition under which the representation (25) can be interpreted as a geometric progression. Thus the cdf  $\Omega_{\gamma<0,\theta}$  corresponding to the chf

$\omega_{\gamma < 0, \theta}$  has the representation as a series of convolutions of the gamma distribution with itself:

$$\Omega_{\gamma < 0, \theta}(x) = \frac{1}{1 - \tilde{A}} \sum_{n=0}^{\infty} \left( \frac{\tilde{A}}{1 - \tilde{A}} \right)^n \bar{F}_{\tilde{\gamma}, \theta}^{*n}(x), \quad (26)$$

where  $\bar{F}_{\tilde{\gamma}, \theta}(x) = 1 - F_{\tilde{\gamma}, \theta}(x)$ , and  $F_{\tilde{\gamma}, \theta}$  is the cdf of gamma distribution with shape parameter  $\tilde{\gamma} = -\gamma$  and scale parameter  $c_{\theta}$ .

- **Case b**  $\gamma \in (0, 1)$ . In this case  $f_{\gamma \in (0, 1), \theta}$  is a chf only when  $\tilde{A} > 0$ , so the representation of the form (26) is not valid. However, as the representation (25) implies that any  $\omega_{\gamma, \theta}$  is a chf as soon as  $f_{\gamma, \theta}$  is a chf, it follows that  $\omega_{\gamma \in (0, 1), \theta}$  is the chf (provided that  $\tilde{A} > 0$ ), and the corresponding r.v. is a geometric analogue of exponentially transformed one-sided stable r.v., corresponding to the chf  $f_{\gamma \in (0, 1), \theta}$ .
- **Case d**  $\gamma \in (1, 2)$ . Recall the representation (27) meaning that  $f_{\gamma \in (1, 2), \theta}$  is a chf linked via the transformation  $\int [\phi_{(2, 2\tilde{c}_{\theta})}(t)]^x p_{\tilde{\gamma}, \theta}(x) dx$  with the pdf  $p_{\tilde{\gamma}, \theta}$  of Case **b** ( $\tilde{\gamma} \in (0, 1)$ ) and with the chf  $\phi_{(2, 2\tilde{c}_{\theta})}$  of the normal distribution (with mean and variance  $(2, 2\tilde{c}_{\theta})$ ). Then  $\omega_{\gamma \in (1, 2), \theta}$  given by (25) is also a chf, the geometric analogue of  $f_{\gamma \in (1, 2), \theta}$ , provided that  $\tilde{A} > 0$  and additionally  $c_{\theta} < 0$ .
- **Case e**  $\gamma = 2$ . As chf  $f_{\gamma=2, \theta}$  corresponds to the Normal r.v.,  $\omega_{\gamma=2, \theta}$  is the chf of the geometric analogue of the Normal r.v.

Check that this case can also be interpreted in the way similar to the Case **b**, so that  $\omega_{\gamma=2, \theta}$  can be viewed as an exponentially transformed Laplace r.v.

- **Case f**  $\gamma > 2$ . This is a new case in our framework, since  $f_{\gamma > 2, \theta}(t)$  does not correspond to a proper probability distribution (in the representation  $\int [e^{P(t)}]^x p_{\theta}(x) dx$  that we used in Case **b** and Case **d**, the polynomial  $P(t)$  can not have a degree greater than 2, due to Marcinkiewicz's theorem [12]).

However,  $\omega_{\gamma > 2, \theta}(t) = (1 - \ln f_{\gamma > 2, \theta}(t))^{-1}$  is a proper chf. That could be seen if we assume  $\gamma \in (2, 4)$ , denote  $\tilde{\gamma} := \gamma/2$  and  $a := 1/(1 - \tilde{A})$  and consider

$$\omega_{\gamma \in (2, 4), \theta}(t) = \frac{1}{1 - \tilde{A}(1 - itc_{\theta})^{\gamma}} = \frac{1}{1 + \tilde{A}[(1 - itc_{\theta})^{\tilde{\gamma}}]^2} = \frac{a}{1 - (1 - a)[(-izc_{\theta})^{\tilde{\gamma}}]^2}$$

with  $z = t + i/c$ . Clearly, the r.h.s. of the above can be represented as  $\mathcal{L}_L(-\log f_{\tilde{\gamma}}^{stab}(z))$  where  $f_{\tilde{\gamma}}^{stab}(z) = \exp\{A[-(-izc_{\theta})^{\tilde{\gamma}}]\}$  is the chf of one-sided stable random variable, as  $\mathcal{L}_L(u) = 1/(1 - b^2 u^2)$  is the Laplace transform of the Laplace distribution. Clearly  $\omega_g(z) := 1/(1 - (1 - a)[(-izc_{\theta})^{\tilde{\gamma}}]^2)$  is a chf of geometric analogue of one-sided stable r.v. (provided that  $(a - 1) > 0$ ), and since  $\omega_g(-i/c) = 1/a$ , then  $\omega_{\gamma \in (2, 4), \theta}(t) = \omega_g(z - i/c)/\omega_g(-i/c)$ . Then  $\omega_{\gamma \in (2, 4), \theta}$  is itself a chf. Note that the condition  $(1 - a) > 0$  means that it should be that  $\tilde{A} > 1$ .

When  $\gamma > 4$ , we can argue analogously. Note that the r.v. corresponding to the chf  $\omega_{\gamma > 2, \theta}$  is nevertheless **not** geometric stable, nor it is geometrically infinitely divisible.

## 5 Summary, interpretation and further extensions of the model

### 5.1 Summary of properties

As the classification of different cases with respect to the ranges of  $\gamma$  spreads out to  $f_\gamma$ ,  $f_{\gamma,\theta}$  and  $\omega_{\gamma,\theta}$ , it might be useful to summarize the relevant information in a summary table.

	Case <b>a</b> $\gamma < 0$	Case <b>b</b> $\gamma \in (0, 1)$	Case <b>d</b> $\gamma \in (1, 2)$	Case <b>f</b> $\gamma > 2$
$f_\gamma$	$\exp \{ \bar{A} [\hat{\gamma}(t) - 1] \}$ • $A < 0$	with $z := t + i/c$ : $\exp \{ -[-izc]^\gamma A \}$ • $A > 0$	with $z := t + i/c$ : $\exp \{ -[(-izc)^2]^\gamma A \}$ • $A > 0$	—
$f_{\gamma,\theta}$	$e^{-\tilde{A}[\hat{\gamma}(t)-1]}$ • $\tilde{A} < 0$	with $z := t + i/c$ : $\exp \{ -[-izc_\theta]^\gamma \tilde{A} \}$ • $\tilde{A} > 0$	with $z := t + i/c_\theta$ : $\exp \{ -[(-izc_\theta)^2]^\gamma \tilde{A} \}$ • $\tilde{A} > 0$	—
Geometr. $\omega_{\gamma,\theta}$	$\frac{1}{\alpha} \sum_{n=0}^{\infty} \alpha^n f_\gamma^n$ • $\tilde{A} < 0$	$(1 - \ln f_{\gamma \in (0,1),\theta}(t))^{-1}$ • $\tilde{A} > 0$	$(1 - \ln f_{\gamma \in (1,2),\theta}(t))^{-1}$ • $\tilde{A} > 0$	$\frac{1-b}{1-b[(-izc_\theta)^{\tilde{\gamma}}]^2}$ • $\tilde{A} > 1$

### 5.2 Interpretation in terms of a subordinated Levy process

While in Section 4 a new class was introduced whose chf  $\phi$  is linked with the chf  $f_{\gamma,\theta}$  through  $\phi(t) = \mathcal{L}(-\ln f_{\gamma,\theta}(t))$ , the chf  $f_{\gamma,\theta}$  itself, for  $1 < \gamma < 2$ , can be represented in the above form. Indeed, according to Case **d** of Paragraph 2.1, we have the following mixture representation for  $f_{\gamma \in (1,2)}$

$$f_{\gamma \in (1,2)}(t) = \int [\phi_{(2,2\bar{\epsilon})}(t)]^x p_{\tilde{\gamma} \in (0,1)}(x) dx = \int e^{x \ln \phi_{(2,2\bar{\epsilon})}(t)} p_{\tilde{\gamma} \in (0,1)}(x) dx = \mathcal{L}_{\tilde{\gamma}}(-\ln \phi_{(2,2\bar{\epsilon})}(t)), \quad (27)$$

and the same is valid for  $f_{\gamma \in (1,2),\theta}$  as well.

Recall that a r.v.  $S$  with chf of the form  $\mathcal{L}_N(-\ln f_X(t))$  can be interpreted as a random sum (with random but finite number of summands), if the Laplace transform  $\mathcal{L}_N$  corresponds to a discrete random variable (a realization of a counting process). One of the most celebrated examples is the Poisson process whose Laplace transform is  $\exp \{ \lambda (e^{-t} - 1) \}$ ,

so that the chf  $\exp \{ \lambda (\phi(t) - 1) \}$  corresponds to the compound Poisson r.v.  $S = \sum_{i=1}^{N(t)} X_i$ .

According to (13), in our case the LT of the r.v. associated with the counting process is  $\mathcal{L}_N(u) = \exp \{ A [1 - (1 + cu)^{\tilde{\gamma}}] \}$  with  $0 < \tilde{\gamma} < 1$ . Clearly, it corresponds to a *continuous* r.v.; moreover, as discussed in Case **b**, the chf  $f_{\tilde{\gamma} \in (0,1)}$  of this r.v. is obtained from the one-sided stable chf by a complex shift, s.t. the corresponding r.v. relates to the exponential transformation of one-sided stable density. Hence the chf (27) cannot be interpreted as a

chf of a compound r.v. in its usual form  $\sum_{i=1}^{N(t)} X_i$ .

However, the corresponding r.v. can be interpreted in terms of continuous-time analogues of random sums — the increments of a *subordinated Levy process*. For the chf of any

Levy process  $X_t$  on  $\mathbb{R}$  we have (see e.g. [2])

$$\mathbf{E} [e^{izX_t}] = e^{t\psi_X(u)}, \quad z \in \mathbb{R}$$

where  $\psi_X$ , usually called the *characteristic exponent* of the Levy process, is defined via  $\phi_X(u) = e^{\psi_X(u)}$ , with  $\phi_X$  being the chf of the r.v.  $X_1$ .

Furthermore, for a subordinated Levy process  $Y_t = X_{S(t)}$  with subordinator  $S(t)$  (which can be viewed as a random compression of time) we have

$$\mathbf{E} [e^{izY_t}] = e^{t \cdot l_S(\psi_X(z))}, \quad z \in \mathbb{R}, \quad (28)$$

where  $l_S(z)$  defined via  $\mathbf{E} [e^{uS(t)}] = e^{t \cdot l_S(u)}$  is called the *Laplace exponent* of  $S$ . For the chf of the increment  $Y_1$  in unit time, (28) turns into  $e^{l_S(\psi_X(z))}$ .

Recall that our chf  $f_{\gamma \in (1,2)}$ , according to (27), can be represented as

$$f_{\gamma \in (1,2)}(z) = \mathcal{L}_{\tilde{\gamma}}(-\ln \phi_{(2,2\bar{\tau})}(z)) = \exp \left\{ A \left[ 1 - (1 - \ln \phi_{(2,2\bar{\tau})}(z))^{\tilde{\gamma}} \right] \right\} = e^{l_{\tilde{\gamma}}(\ln \phi_{(2,2\bar{\tau})}(z))},$$

where  $l_{\tilde{\gamma}}$  is the Laplace exponent of the Levy process whose increments have the chf  $f_{\tilde{\gamma} \in (0,1)}$ , i.e.  $f_{\tilde{\gamma} \in (0,1)}(iu) = \mathcal{L}_{\tilde{\gamma}}(-u) = e^{l_{\tilde{\gamma}}(u)}$ . Therefore  $f_{\gamma \in (1,2)}$  can be interpreted as a chf of the r.v.  $Y_1$ , i.e. of the increments of subordinated Levy process  $Y_t = X_{S(t)}$  in unit time  $t = 1$ . The increments of the underlying process  $X_t$  have normal distribution with parameters  $(2, 2\bar{\tau})$ , while the increments of the subordinator  $S(t)$  are distributed according to the exponentially transformed one-sided Levy distribution whose chf is given by  $f_{\tilde{\gamma} \in (0,1)}$ .

A possible way to extend the model is to use **any** relevant chf as the component  $\phi$  in the representation (27). That would also lead to an interpretation of the corresponding distribution as a distribution of the increments of a subordinated Levy process  $Y_t = X_{S(t)}$ , with the same subordinator  $S(t)$  but with a different underlying process  $X_t$ .

## A Derivation of the chf in the explicit form

The function  $f$  to be found satisfies the balance equations

$$f_{\theta}(t) = \frac{f(t - i\theta)}{f(-i\theta)} \quad \text{and} \quad f_{\theta}(t) = f^{\alpha(\theta)}(\beta(\theta)t) \quad \forall t \in \mathbb{R}, \quad \theta \in [a, b], \quad 0 \in [a, b].$$

In terms of the log-function  $g(t) = \log f(t)$ , it turns into

$$g(t - i\theta) = g(-i\theta) + \alpha(\theta)g(\beta(\theta)t).$$

In order to prove that  $g$  can be represented in the form (5), let us differentiate  $g(t - i\theta)$  w.r.t. to  $t$  and to  $\theta$ . First differentiate it w.r.t. to  $t$

$$g'(t - i\theta) = \alpha(\theta) \cdot \beta(\theta) \cdot g'(\beta(\theta)t). \quad (29)$$

Considering a limit w.r.t  $\theta \rightarrow 0$  and assuming that  $\alpha(0)\beta(0) \neq 0$ , we get

$$g'(t) = Ag'(bt), \quad (30)$$

where  $b := \beta(0)$  and  $A := \alpha(0)\beta(0)$ . Two major cases are possible:  $b \neq 1$  and  $b = 0$ . Let us start with the first one.

$b \neq 1$

Without loss of generality, assume that  $|b| < 1$ . Let us differentiate (30) further  $k$  times

$$g^{(k+1)}(t) = A \cdot b^k g^{(k+1)}(bt),$$

and consider it at  $t = 0$

$$g^{(k+1)}(0) = A \cdot b^k g^{(k+1)}(0).$$

Considering a limit w.r.t  $k \rightarrow \infty$ , we see that  $1 < Ab^k \rightarrow 0$ , which means that

$$g^{(k+1)}(0) = 0 \quad \text{when} \quad k > K_0 \quad (\text{for some large enough } K_0).$$

It follows from the latter that  $g(t)$  is some polynomial  $g(t) = P(t)$ , so that the chf  $f(t)$  is the exponent of a polynomial  $f(t) = e^{P(t)}$ . Due to Marcinkiewicz's theorem [12], we can conclude that the degree of  $P$  should not be higher than 2, e.g. it is a quadratic polynomial.

In other words,  $b \neq 1$  in (30) corresponds to the case when  $f(t)$  is the chf of the Normal distribution considered in Section 2.1 as Case e.

$b = 1$

Specifically,  $b = 1$  means that not just  $\beta(0) = 1$  but also  $\alpha(0) = 1$ , as follows from (29). Denoting  $\Lambda(\theta) := \alpha(\theta)\beta(\theta)$  and keeping in mind that  $\Lambda(0) = 1$ , let us now differentiate (29) w.r.t to  $\theta$  which gives

$$-ig''(t - i\theta) = \Lambda'(\theta)g'(\beta(\theta)t) + \Lambda(\theta)\beta'(\theta)g''(\beta(\theta)t)t$$

and considering a limit w.r.t  $\theta \rightarrow 0$ , we get

$$-ig''(t) = \Lambda'(0)g'(t) + t\beta'(0)g''(t),$$

which leads to

$$-(t\beta'(0) + i)g''(t) = \Lambda'(0)g'(t)$$

Denoting  $u(t) := g'(t)$ , we get an equation

$$\frac{u'(t)}{u(t)} = \frac{-\Lambda'(0)}{(i + t\beta'(0))} \quad (31)$$

The solution of this equation gives

$$\ln u(t) = -c_1 \ln(i + t\beta'(0)) + c_2 \iff u(t) = \frac{c}{(t\beta'(0) + i)^{c_1}} \quad (\text{where } c := e^{c_2}),$$

i.e.  $g'(t) = c(t\beta'(0) + i)^{-c_1}$ , which means that

$$g(t) = \frac{c}{(1 - c_1)\beta'(0)} (\beta'(0)t + i)^{1-c_1} + c_2.$$

Denote  $1 - c_1 =: \gamma$  and, in order to express the constant  $c_2$  explicitly, consider the above at  $t = 0$

$$\frac{c}{(1 - c_1)\beta'(0)} i^\gamma = -c_2,$$

so that we can denote  $A_1 := \frac{c}{(1-c_1)\beta'(0)}$  and rearrange to get

$$g(t) = A_1 [(\beta'(0)t + i)^\gamma - i^\gamma].$$

With two more notations  $A := -A_1 i^\gamma$  and  $c := \beta'(0)$ , after rearrangement it finally turns into

$$g(t) = A [1 - (1 - ict)^\gamma].$$

Throughout the derivation, we assumed that  $\beta'(0) \neq 0$ . Let us now consider the opposite case.

$$\beta'(0) = 0$$

It easy to check that this corresponds to a degenerate case, as follows from (31)

$$\frac{u'(t)}{u(t)} = -const = c \iff \ln u(t) = ct + c_2,$$

which implies that  $u(t) = \tilde{c}e^{ct}$ .

Eventually, the general form of the function  $g$  in question is

$$g(t) = A [1 - (1 - itc)^\gamma].$$

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